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The Proximinality of the Centre of a C^*-Algebra

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It is shown that if A is a unital C^*-algebra then $Z(A)$, the centre of A, is a proximinal subspace. In other words, for each $a \in A$ there exists $z \in Z(A)$ such that $\|a - z\|$ is equal to the distance from a to $Z(A)$. © 1997 Academic Press

INTRODUCTION

A subspace X of a Banach space Y is said to be *proximinal* if for each $y \in Y$ there exists $x \in X$ such that $\|y - x\|$ is equal to the distance from y to X; see [5] for background information. There is no reason, in general, to expect subspaces to be proximinal, so it is always interesting to discover conditions which force proximinality. For example, a simple compactness argument shows that each finite-dimensional subspace is proximinal. For C^*-algebras, it is known that every closed, two-sided ideal is proximinal [1, Theorem 4.3], and Victor Shulman has pointed out that [3, 3.9] implies that every closed one-sided ideal is also proximinal.

If A is an abelian C^*-algebra with an identity and B is a C^*-subalgebra of A containing the identity then B is proximinal [7, Theorem 2]. Mazur had shown earlier, see [8, 7.5.6], that B_{sa}, the self-adjoint part of B, is proximinal in A_{sa}. Now suppose that A is a noncommutative C^*-algebra and a is a self-adjoint element of A. Let B be the C^*-subalgebra of A generated by a and $Z(A)$, the centre of A. The distance from a to $Z(A)_{sa}$ is the same whether computed in A or B, and Mazur's result says that $Z(A)_{sa}$ is a proximinal subspace of B_{sa}. It follows that $Z(A)_{sa}$ is a proximinal subspace of A_{sa}. This result was obtained, by essentially the same method, in [9]. It raises the question of whether $Z(A)$ itself is proximinal in A. In this paper we show that it is.

We now introduce some of the definitions that we need. Let A be a C^*-algebra with an identity. Recall from [2, p. 351] that the *Glimm* ideals of A are the ideals of A generated by the maximal ideals of $Z(A)$ (Glimm ideals are automatically closed by the Cohen Factorization Theorem). The

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sum of two maximal ideals of $Z(A)$ contains the identity, from which it is clear that the Glimm ideals of A are in one-to-one correspondence with the maximal ideals of $Z(A)$. The set of Glimm ideals of A is denoted Glimm(A), and equipped with the topology from the maximal ideal space of $Z(A)$, so that Glimm(A) is a compact Hausdorff space, homeomorphic to the maximal ideal space of $Z(A)$. Thus we can identify $Z(A)$ with the algebra of continuous complex-valued functions on Glimm(A). Furthermore, for each $a \in A$ the map $G \to \|a_G\|$ $(G \in \text{Glimm}(A))$ is upper semicontinuous on Glimm(A); see [2, p. 351; 4, Lemma 9]. (Here a_G denotes the canonical image of a in the quotient C^*-algebra A/G). Each primitive ideal of A intersects $Z(A)$ in a maximal ideal, and therefore contains a (unique) Glimm ideal of A. This implies that the Glimm ideals of A have zero intersection.

We use the following two theorems. The first is a very useful result from [11]. If A is a unital C^*-algebra and $a \in A$, let $\lambda(a)$ be the nearest scalar to a.

THEOREM 1. *Let A be a C^*-algebra with an identity and let $a \in A$. Then for any $\lambda \in \mathbf{C}$*

$$\|a - \lambda(a)\|^2 + |\lambda - \lambda(a)|^2 \leqslant \|a - \lambda\|^2.$$

The next theorem was proved in [10, 2.3].

THEOREM 2. *Let A be a C^*-algebra with an identity and let $a \in A$. Then the distance from a to the centre of A is equal to*

$$\sup\{\|a_G - \lambda(a_G)\| : G \in \text{Glimm}(A)\}.$$

Recall that a set-valued function F from a topological space X into the set of subsets of a topological space Y is said to be *lower semicontinuous* if for each open subset U of Y the set $\{x \in X : F(x) \cap U \neq \varnothing\}$ is open in X.

THEOREM 3. *Let A be a C^*-algebra with an identity. Then the centre of A is a proximinal subspace.*

Proof. Let $a \in A$. We may assume that the distance from a to $Z(A)$ is strictly positive, α, say. We shall assume, for convenience, that $\alpha = 1$, although we will continue to call it α to show where it comes from. For each $G \in \text{Glimm}(A)$ define X_G by $X_G = \{\lambda \in \mathbf{C} : \|a_G - \lambda\| \leqslant \alpha\}$. It is easy to see that X_G is a closed convex set, nonempty by Theorem 2. We prove that the function $G \to X_G$ is a lower semicontinuous set-valued map from Glimm(A) into the set of subsets of $\mathbf{C}$.

Let U be any open subset of $\mathbf{C}$, and suppose that $G \in \text{Glimm}(A)$ with $X_G \cap U$ nonempty, that is, there exists $\lambda \in U$ such that $\|a_G - \lambda\| \leqslant \alpha$. Let

$\varepsilon \in \mathbf{R}$, with $0 < \varepsilon < 1$, such that the ball of radius ε, centred on λ, is contained in U. Let $N = \{H \in \mathrm{Glimm}(A) : \|a_H - \lambda\| < \alpha + \varepsilon^3/8\}$. Then N is an open neighbourhood of G, by the upper semicontinuity mentioned above. Let $H \in N$ and set $\beta = \alpha - \|a_H - \lambda(a_H)\|$. Then $1 \geqslant \beta \geqslant 0$, by Theorem 2. Theorem 1 implies that

$$\|a_H - \lambda(a_H)\|^2 + |\lambda - \lambda(a_H)|^2 \leqslant \|a_H - \lambda\|^2 < (\alpha + \varepsilon^3/8)^2,$$

so

$$|\lambda - \lambda(a_H)|^2 < (\alpha + \varepsilon^3/8)^2 - (\alpha - \beta)^2 = \varepsilon^3/4 + \varepsilon^6/64 + 2\beta - \beta^2 \qquad (*)$$

(since $\alpha = 1$). We consider two cases:

(i) $\beta < \varepsilon^2/4$. It follows from $(*)$ above that $|\lambda - \lambda(a_H)|^2 < \varepsilon^3/4 + \varepsilon^6/64 + \varepsilon^2/2 < (49/64)\varepsilon^2$. Hence $|\lambda - \lambda(a_H)| < (7/8)\varepsilon$, so $\lambda(a_H) \in X_H \cap U$.

(ii) $\beta \geqslant \varepsilon^2/4$. Set $\mu = (1 - \varepsilon/2)\lambda + (\varepsilon/2)\,\lambda(a_H)$. Then

$$\begin{aligned} \|a_H - \mu\| &= \|(1 - \varepsilon/2)(a_H - \lambda) + (\varepsilon/2)(a_H - \lambda(a_H)\| \\ &\leqslant (1 - \varepsilon/2)\,\|a_H - \lambda\| + (\varepsilon/2)\,\|a_H - \lambda(a_H)\| \\ &< (1 - \varepsilon/2)(\alpha + \varepsilon^3/8) + (\varepsilon/2)(\alpha - \beta) \\ &\leqslant (1 - \varepsilon/2)(\alpha + \varepsilon^3/8) + (\varepsilon/2)(\alpha - \varepsilon^2/4) \\ &= \alpha - \varepsilon^4/16 < \alpha, \end{aligned}$$

so $\mu \in X_H$. Furthermore, $|\lambda - \mu| = (\varepsilon/2)\,|\lambda - \lambda(a_H)|$, and $2\beta - \beta^2 \leqslant 1$ for all β, so it follows from $(*)$ above that

$$\begin{aligned} |\lambda - \lambda(a_H)|^2 &< \varepsilon^3/4 + \varepsilon^6/64 + 2\beta - \beta^2 \\ &< 1/4 + 1/64 + 1 = 81/64. \end{aligned}$$

Hence $|\lambda - \mu| < (\varepsilon/2)(9/8) < \varepsilon$. Thus $\mu \in X_H \cap U$.

This shows that the map $G \to X_G$ is lower semicontinuous. Thus Michael's Selection Theorem [6] implies that there is a continuous function f on $\mathrm{Glimm}(A)$ such that $f(G) \in X_G$ for each $G \in \mathrm{Glimm}(A)$. Let $z \in Z(A)$ such that $z_G = f(G)$ for each G. Since the Glimm ideals of A have zero intersection,

$$\begin{aligned} \|a - z\| &= \sup\{\|a_G - z_G\| : G \in \mathrm{Glimm}(A)\} \\ &= \sup\{\|a_G - f(G)\| : G \in \mathrm{Glimm}(A)\} \\ &\leqslant \alpha. \end{aligned}$$

Hence $\|a - z\|$ is equal to the distance from a to $Z(A)$. Q.E.D.

Remarks. (i) The distance from a to $Z(A)$ is of interest because it is related to the norm of the inner derivation induced by a; see [9, 10]. Since a and b induce the same inner derivations if and only if $a - b \in Z(A)$, Theorem 3 implies that for each inner derivation on a unital C^*-algebra there exists an element of minimal norm implementing the derivation.

(ii) The method of proof also shows that each C^*-subalgebra B of $Z(A)$ which contains the identity is proximinal. One simply replaces the Glimm ideals by the family of ideals of A generated by the maximal ideals of B. The proofs of Theorem 2 and Theorem 3 go through unchanged.

(iii) Reference [3, 3.12] gives an example of a unital C^*-algebra A with a nonproximinal abelian C^*-subalgebra B. In fact, B_{sa} is not proximinal in A_{sa}.

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